COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **22** (3) (2002) 345–359

SMALL UNIVERSAL GRAPHS FOR BOUNDED-DEGREE PLANAR GRAPHS

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Received July 8, 1998

For all positive integers N and k, let $\mathcal{G}(N,k)$ denote the family of planar graphs on N or fewer vertices, and with maximum degree k. For all positive integers N and k, we construct a $\mathcal{G}(N,k)$-universal graph of size $O_k(N)$. This construction answers with an explicit construction the previously open question of the existence of such a graph.

1. Introduction

Given a family $\mathcal{G}$ of undirected, simple graphs, a graph H is defined to be $\mathcal{G}$*-universal* if, for each $G \in \mathcal{G}$, the graph H contains a subgraph that is isomorphic to G. For example, if $\mathcal{G}_N$ is the family of graphs on N or fewer vertices, then the complete graph K_N on N vertices in $\mathcal{G}_N$-universal. Universal graphs have importance in a variety of practical applications. For example, as discussed in [2], universal graphs are of interest to chip manufacturers. It is very expensive to design computer chips, but relatively inexpensive to make copies of a computer chip with the same design. This encourages manufacturers to make their chip configurable, in the sense that the entire chip is prefabricated except for the last layer, and then a final layer of metal is then added corresponding to the circuitry of a customer's specific application. Hence, most of the design costs can be amortized over many customers. We may view the circuitry of a computer chip as a graph, and we may also model the problem of designing such chips with fewer wires that are configurable for a particular family of applications as designing smaller universal

Mathematics Subject Classification (2000): 05C35

* Supported by NSF grant CCR98210-58 and ARO grant DAAH04-96-1-0013.

graphs for a particular family of graphs. Thus, with these applications in mind, we note that is often desirable to find a small $\mathcal{G}$-universal graph given a family $\mathcal{G}$ of graphs.

In this paper we study the family of bounded-degree planar graphs. More specifically, for every two positive integers N and k, let $\mathcal{G}(N,k)$ denote the family of planar graphs on N vertices and with maximum degree k. Bhatt, Chung, Leighton, and Rosenberg [1] constructed a $\mathcal{G}(N,k)$-universal graph of size $\theta_k(N \lg N)$, where we define the *size* of a graph H to be the quantity $|V(H)|+|E(H)|$. They then posed the question to whether there exists asymptotically smaller $\mathcal{G}(N,k)$-universal graphs. Here we settle this open question by constructing $\mathcal{G}(N,k)$-universal graphs of size $O_k(N)$, which is of course asymptotically optimum for when k is fixed.

We now describe the layout of this paper. In § 2 we construct a graph $H(N,3)$ of size $O(N)$, and then prove in § 3 that $H(N,3)$ is $\mathcal{G}(N,3)$ universal. We then show in § 4 how to modify the construction of $H(N,3)$ to obtain, for general k, the graph $H(N',k)$ that has size $O_k(N')$, and is $\mathcal{G}(N',k)$-universal.

We now specify notation that we will use. First, for any undirected graph G, and any two vertices x and x' in G, we often write the number of edges in the shortest path from x to x' as $d_G(x,x')$. Finally, for all positive n, we write $\log_2 n$ as $\lg n$.

A preliminary version of this paper appeared in [3].

2. The construction of small $\mathcal{G}(N,3)$-universal graphs

In this section we present the construction of a graph $H(N,3)$, for all positive integers N. The construction of $H(N,3)$ is a nontrivial modification of the graph $\tilde{H}(N,3) = \tilde{H}$ presented in [1], which we now briefly describe. The graph $\tilde{H}$ is constructed from T, and T from $\hat{T}$, where $\hat{T}$ is an N-vertex complete binary tree, and T is formed from $\hat{T}$ by linking each vertex v with each ancestor y that are of distance $O(1)$ from each other in $\hat{T}$. Then $\tilde{H}$ is formed from T by first replacing each vertex $y \in T$ in $\hat{T}$ with a clique on a set $\tilde{V}_y$ of $\theta(2^{\frac{h(y)}{2}})$ vertices, where $h(y)$ is the height of y in $\hat{T}$, and then replace each edge $\{u,v\}$ in T with a complete bipartite graph between $\tilde{V}_u$ and $\tilde{V}_v$. So each vertex in $\tilde{V}_y$ has degree $\theta(2^{h/2})$ in $\tilde{H}$. And as there are $\theta(N \times 2^{-h})$ vertices y in T where $h(y) = h$ for each positive integer $h \leq \lg N$, the size of the resulting graph $\tilde{H}$ is $\Omega(N \lg N)$.

To construct $H(N,3)$, we do the following. Instead of replacing each vertex $y \in T$ with a clique on $\theta(2^{h/2})$ vertices, where h is the height of y in $\hat{T}$, we replace y with a significantly sparser graph on a set V_y of about the same number of vertices. And instead of putting a complete bipartite graph between $\tilde{V}_u$ and $\tilde{V}_v$ if u and v are adjacent in T, we put a significantly sparser bipartite graph between V_u and V_v if $d_T(u,v)$ is $O(1)$. To elaborate, we first replace y with a complete binary tree T_y on $\theta(h^2)$ vertices (the T_y's are vertex-disjoint). Then we interconnect the T_y's in a specified way to get a bounded-degree graph T'. Next, we replace each vertex $y' \in T_y$ with a clique on a set $V'_{y'}$ of $\theta(h^{-2} \times 2^{h/2})$ vertices. Finally, for every two vertices u' and v' in T' where $d_{T'}(u',v')$ is $O(1)$, we put a complete bipartite graph between $V'_{y'}$ and $V'_{w'}$. The resulting graph is $H(N,3)$, which we will often write as H.

So for each vertex $y \in Y$ of height h in T, there are $\theta(2^{h/2})$ vertices in $V_y = \cup_{y' \in T_y} V_{y'}$, just as there are in $\tilde{V}_y$. However, each vertex in V_y has degree only $O(2^{h/2} \times h^{-2})$ in H, so the size of H is $O(N)$.

We specify the construction of T' and H precisely below.

Construction of $H(N,3)$. We construct T' from $\hat{T}$ in steps (1)–(3), where $\hat{T}$ is a complete binary tree on N vertices. Then we construct $H(N,3)$ from T' in step (4).

(1) Interconnect every vertex v of $\hat{T}$ with each ancestor y that satisfies $d_{\hat{T}}(v,y) \leq 11$. Call the resulting graph T.

(2) For each vertex y in T, let T_y be a complete rooted binary tree of height $2\lg h(y)$, where $h(y)$ is the height of y in $\hat{T}$ (so T_y has $\theta(h^2(y))$ vertices), such that the T_y's are *vertex-disjoint*. For each vertex $y \in T$, and each interior vertex $y' \in T_y$, let $\sigma_1(y')$ and $\sigma_2(y')$ denote the left and right child of y', respectively.

(3) The vertex-set of T' is $\cup_{y \in T} V(T_y)$, and for each vertex $y \in T$, the induced subgraph of T' on $V(T_y)$ is T_y. So we next specify how to add edges between the T_y's to get T'. We interconnect the T_y's in the following fashion.

(A) We interconnect every vertex in the top 11 levels (i.e., the $2^{11}-1$ vertices within distance 11 from the roots) of T_y with every vertex in the top 11 levels of T_w if and only if y and w are adjacent in T.

(B) Next, we proceed recursively in the following fashion. Let w and y be adjacent vertices in T, and let $y' \in T_y$ and $w' \in T_w$ be interconnected. Then for each $\iota \in \{1,2\}$, interconnect $\sigma_\iota(y')$ with $\sigma_\iota(w')$ (if they exist).

The resulting graph is T'.

(4) For each vertex $y' \in T'$, let $V'_{y'}$ be a set of of $O(h^{-2}(y) \times 2^{\frac{h(y)}{2}})$ vertices such that the $V'_{y'}$'s are disjoint, where y is the vertex in T such that $y' \in T_y$. Then let $H(N,3) = H$ be the following graph on $\cup_{v' \in T'} V'_{y'} = V(H)$. Let u''

and v'' be vertices in $V(H)$, and let u' and v' be the vertices in T' where $u'' \in V'_{u'}$, and $v'' \in V'_{v'}$. Then u'' and v'' are adjacent in H if and only if either (1) $u' = v'$, or (2) the quantity $d_{T'}(u', v')$ is no larger than 18. ∎

We will now show that the size of $H(N,3)$ is $O(N)$. For each vertex $u' \in T'$, define $h(u')$ to be the quantity $h(u)$, where u is the vertex in T where $u' \in T_u$. Next, let T'' be the graph on $V(T')$ where u' and v' are adjacent in T'' if and only if $d_{T'}(u', v')$ is no larger than 18. Note that T' has bounded-degree, and so does T'', and there is an edge in H between $V'_{u'}$ and $V'_{v'}$ only if u' and v' are adjacent in T'', or if $u' = v'$. Now let u' and v' be two distinct adjacent vertices in T'' where $h(v') \geq h(u')$. Then the number of edges in H between $V_{u'}$ and $V'_{v'}$ is $O(2^{h(v')} \times h^{-4}(v'))$, and the number of edges in H with both endpoints in $V_{v'}$ is at most $O(2^{h(v')} \times h^{-4})$. So $E(H)$ satisfies

$$|E(H)| \leq \sum_{h} \sum_{v' \in V(T'); h(v') = h} O(Dh^{-4} \times 2^h),$$

where D is the maximum degree of T''. But the number of vertices $v' \in T'$ such that $h(v') = h$ is $O(h^{-2} \times N/2^h)$. Indeed, there are $N/2^h$ vertices $u \in T$ of height h in $\hat{T}$, and $\theta(h^2)$ vertices in T_u, for each such u. So it follows that

$$|E(H)| \leq \sum_{h} O(h^2 \times N/2^h) \times O(Dh^{-4} \times 2^h),$$

which is $O(DN) = O(N)$, since the maximum degree D of T'' is $O(1)$. Thus the size of $H(N,3)$ is $O(N)$.

So we prove this theorem in the next section.

Theorem 2.1. *The graph $H(N,3) = H$ is $\mathcal{G}(N,3)$-universal.*

3. Embedding graphs in $\mathcal{G}(N,3)$ into $H(N,3)$

For any $G \in \mathcal{G}(N,3)$ into $H(N,3) = H$, suppose that there exists a function $\chi_G : V(G) \mapsto V(H)$ that satisfies the following properties.

(I$''$) If x and x' are adjacent vertices in G, then $\chi_G(x)$ and $\chi_G(x')$ are adjacent vertices in H.

(II$''$) χ_G maps at most 1 vertex of G onto v'' for each vertex $v'' \in H$.

Then each $G \in \mathcal{G}(N,3)$ is isomorphic to a subgraph of H, and therefore, H is $\mathcal{G}(N,3)$-universal, and so Theorem 2.1 is proved. However, for each

$G \in \mathcal{G}(N,3)$, if there exists an χ_G that satisfies (I$''$) and (II$''$) if there exists a $g_G : V(G) \mapsto V(H)$ that satisifies the following properties.

(I') If x and x' are adjacent vertices in G, then $d_{T'}(g_G(x), g_G(x'))$ is no larger than 18.

(II') g_G maps $O(h^{-2}(v) \times 2^{\frac{h(v)}{2}})$ vertices of G onto v' for each vertex $v' \in T'$, where v is the vertex in T such that $v' \in T_v$.

Thus, we make the following observation.

Lemma 3.1. *If, for each $G \in \mathcal{G}(N,3)$, there exists a $g_G : V(G) \mapsto V(T')$ that satisfies (I') and (II'), then H is $\mathcal{G}(N,3)$-universal, and Theorem 2.1 is proved.* ∎

So for the rest of this section, we prove the existence of such a g_G. We use the following. Bhatt et. al. [1] give, for each $G \in \mathcal{G}(N,k)$, and embedding $f_G : V(G) \mapsto T$ such that

(I) If x and x' are adjacent vertices in G, then $f_G(x)$ and $f_G(x')$ are either identical or adjacent vertices in T.

(II) f_G maps at most $O(2^{\frac{h(v)}{2}})$ vertices of G onto v for each vertex $v \in T$.

So to obtain g_G, we refine f_G. In other words, for all vertices $x \in V(G)$, if $f_G(x)$ is the vertex v in T, then $g_G(x)$ is a vertex in T_v. We now present some notation that we will use. For each $y' \in T'$, we will often write the set $\{x \in V(G) |\ g_G(x) = y'\}$, or equivalently, the set of vertices of G that g_G maps to y', as $g_G^{-1}(y')$. Similarly, for each vertex $y \in T$, we will often write the set of vertices of G that f maps to y as $f_G^{-1}(y)$. Next, we introduce a function $r : V(T') \mapsto \mathbf{Z}$. Let y' be an arbitrary vertex in T', and let y be the vertex in T where y' in T_y. Then we define $r(y')$ to be the height of y in $\hat{T}$, minus the depth of y' in T_y. So if y' is the root of T_y, then $r(y') = h(y)$; if y' is a child of the root of T_y, then $r(y') = h(y) - 1$, and so on; if y' is a leaf of T_y, then $r(y') = h(y) - 2 \lg h(y)$, since T_y has height $2 \lg h(y)$. So $r(y') \leq \lg N$ for each vertex in T'. Next, let Y^r be the vertices $y \in T$ such that T_y contains a vertex y' such that $r(y') = r$. Then Y^r is the set of vertices $y \in T$ such that $h(y) - 2 \lg h(y) \leq r \leq h(y)$. Finally, let X^r be the set of vertices $x \in G$ such that $f_G(x) \in Y^r$, and let G^r be the induced subgraph of G on X^r. We next describe our procedure for construction g_G from f_G.

Construction of g_G from f_G. We construct g_G in phases $\lg N$, $\lg N - 1, \ldots,$ and so on. By the beginning of phase r, for general r, we have already specified $g_G^{-1}(w')$ for each vertex $w' \in T'$ such that $r(w') \geq r + 1$. We have also specified the set $V(G)_{v'}$ for each vertex v' such that $r(v')=r$, where we define, for each vertex $u'\in T'$, the set $V(G)_{u'}$ to be the set of vertices x of G such that $g_G(x)$ is either u', or a descendant of u' in T_u, where u is the vertex in T such that $u'\in T_u$. (If u' is the root of T then we specify $V(G)_{u'}$ to be $f_G^{-1}(u)$, as we want each vertex in $f_G^{-1}(u)$ to be mapped by g_G to a vertex in T_u.) Then, during phase r, for each vertex $v'\in T'$ such that $r(v')=r$, we partition $V(G)_{v'}$ into 3 sets $g_G^{-1}(v')$, and $V(G)_{\sigma_1(v')}$, and $V(G)_{\sigma_2(v')}$. Then, as each vertex $u'\in T'$ such that $r(u')=r-1$ is either (a) the root of some T_u, or (b) of the form $\sigma_\iota(v')$ for some v' such that $r(v')=r$, we will have specified $V(G)_{u'}$ for u' by the end of phase r.

We now describe phase r, for general r. For each vertex $v'\in T'$ such that $r(v')=r$, we have already specified $V(G)_{v'}$ before. First, for each positive integer c, let C^{r+c} be the set of vertices $x\in V(G)$ where x is adjacent in G to a vertex x' such that $r(g_G(x'))=r+c$. Next, let S^r be a subset of $X^r=V(G^r)$ that satisfies **(A)** and **(B)**, stated next.

(A) $|S^r\cap V(G)_{v'}|\leq O(h^6(v)\times 2^{\frac{h(v)}{4}})$ for each vertex $v'\in T'$ where $r(v')=r$, and where v is the vertex in T such that $v'\in T_v$.

(B) We may write $G^r\setminus S^r=G_1^r\cup G_2^r$, where (a) G_1^r and G_2^r are vertex-disjoint, and there is no edge in G with one endpoint in G_1^r and the other in G_2^r, and (b) each G_ι^r contains no more than half of the vertices of $V(G)_{v'}$, for each $v'\in T'$ such that $r(v')=r$.

(We will prove later that there exists such an S^r.) Next, let v' be a vertex in T' such that $r(v')=r$. If v' is an interior vertex of T_v set

$$g_G^{-1}(v')=V(G)_{v'}\cap(S^r\cup C^{r+17}), \tag{1}$$

and for each $\iota\in\{1,2\}$, let $V(G)_{\sigma_\iota(v')}$ be the set of vertices that are in $V(G)_{v'}\cap V(G_\iota^r)$, and are not in $g_G^{-1}(v')$. On the other hand, if v' is a leaf of T_v then simply set $g_G^{-1}(v')=V(G)_{v'}$. ∎

So the rest of this section is devoted to showing that there indeed exist S^r that satisfy **(A)** and **(B)**, and then that g_G as constructed satisfies (I') and (II').

We now make some observations. Let v' be a vertex in T', and let v be the vertex in T where $v'\in T_v$. Let us also assume for now that there exist S^r that satisfy **(A)** and **(B)**; we will prove that there indeed do by proving Lemma 3.3. Then the set $V(G)_{v'}$ is a subset of $f_G^{-1}(v)$. Also, if $r(v')=r$, then $v\in Y^r$ by definition of Y^r, and so $f_G^{-1}(v)\subseteq X^r$ by definition of X^r. So $V(G)_{v'}\subseteq X^r=V(G^r)$, and thus, the sets $g_G(v')$, $V(G)_{\sigma_1(v')}$, and $V(G)_{\sigma_2(v')}$

partition $V(G)_{v'}$. Thus, each vertex of G gets mapped by g_G to exactly one vertex in T'; in fact, each vertex in $f_G^{-1}(v)$ gets mapped by g_G to a vertex in T_v. So g_G is indeed a function from $V(G)$ to $V(T')$. So if both there exist S^r that satisfy **(A)** and **(B)** and the resulting g_G satisfies both (I') and (II'), then Theorem 2.1 follows. So we prove Lemma 3.3, and then we prove Proposition 3.4, stated below.

Lemma 3.3 establishes that there exists S^r that satisfy **(A)** and **(B)**. The proof of Lemma 3.3 is Theorem 3.2, which follows from the results proved by Lipton and Tarjan [5], combined with the results proved by Goldberg and West [4].

Theorem 3.2. follows from [4] combined with [5]: *Let G be a finite planar graph on M vertices, and let $\mathcal{V}$ be a set of disjoint sets of $V(G)$. Then there exists a subset S of $V(G)$ such that*

(1) *writing $G \setminus S = G_1 \cup G_2$, the graphs G_1 and G_2 are vertex-disjoint, and there is no edge with one endpoint in G_1, and the other in G_2, and each G_i contains no more than half the number of vertices of each $S \in \mathcal{V}$, and*

(2) *the number of vertices in S is no more than $O(|\mathcal{V}|M^{1/2})$.* ∎

So we now use Theorem 3.2 to prove that there exist such S^r.

Lemma 3.3. *For each r, there exists a subset S^r of $X^r = V(G^r)$ that satisfies the following two conditions* **(A)** *and* **(B)**.

(A) $|S^r \cap V(G)_{v'}| \leq O(h^6(v) \times 2^{\frac{h(v)}{4}})$ *for each vertex $v' \in T'$ where $r(v') = r$, and where v is the vertex in T such that $v' \in T_v$.*

(B) *We may write $G^r \setminus S^r = G_1^r \cup G_2^r$, where (a) G_1^r and G_2^r are vertex-disjoint, and there is no edge in G with one endpoint in G_1^r and the other in G_2^r, and (b) each G_ι^r contains no more than half of the vertices of $V(G)_{v'}$, for each $v' \in T'$ where $r(v') = r$.*

Proof. Let T^r be the induced subgraph of T on Y^r. Next, let $T^{r,1}, \ldots, T^{r,m_r}$ be the components of T^r, and for each $i \in \{1, \ldots, m_r\}$, let $X^{r,i}$ be the set of vertices $x \in V(G)$ such that $f_G(x) \in V(T^{r,i})$, or equivalently, let $X^{r,i} = \cup_{y \in V(T^{r,i})} f_G^{-1}(y)$. Then the $X^{r,i}$'s are disjoint. Next, for each $i \in \{1, \ldots, m_r\}$ let $G^{r,i}$ be the induced subgraph of G on $X^{r,i}$. Then, for each such i, let $S^{r,i}$ be the minimum-sized subset of $X^{r,i}$ such that we may write $G^{r,i} \setminus S^{r,i} = G_1^{r,i} \cup G_2^{r,i}$, where (a) $G_1^{r,i}$ and $G_2^{r,i}$ are vertex-disjoint, and there is no edge in G between $G_1^{r,i}$ and $G_2^{r,i}$, and (b) for each vertex $v' \in T'$ such that $r(v') = r$ and $V(G)_{v'}$ intersects $V(G^{r,i})$, each $G_\iota^{r,i}$ contains no more than half of the vertices of $V(G)_{v'}$. Finally, let $\mathcal{T}^r$ be the collection of components of T^r, and let $\mathcal{X}^r$ be the collection of the $X^{r,i}$'s, and then let $\mathcal{S}^r$ be the collection of the $S^{r,i}$'s.

Then, set $S^r = \cup_{i=1}^m S^{r,i}$, and $G_\iota^r = \cup_{i=1}^m G_\iota^{r,i}$ for each $\iota = 1,2$. Then S^r satisfies **(B)**. Indeed, the G_ι^r's are vertex-disjoint because the $X^{r,i}$'s are disjoint. Also, there are no edges between the G_ι^r's. Indeed, if there were, there would be an edge between $V(G_1^{r,i})$ and $V(G_2^{r,i'})$ for two distinct i and i'. This would imply that there are vertices $y \in T^{r,i}$ and $w \in T^{r,i'}$ such that there is an edge in G between $f_G^{-1}(y)$ and $f_G^{-1}(w)$. But this is impossible, since f_G satisfies (I), and y and w are in different components of T^r. So S^r satisfies (a) of **(B)**. Also,$V(G)_{w'}$, for any vertex $w' \in T'$, intersects at most one $X^{r,i} \in \mathcal{X}^r$. Specifically, let w be the vertex in T such that $w' \in T_w$. Then $V(G)_{w'}$ intersects (and is a subset of) $X^{r,i}$ if and only if i is the integer in $\{1,\ldots,m_r\}$ such that $w \in T^{r,i}$. Indeed, $V(G)_{w'} \subseteq f_G^{-1}(w)$. But $f_G^{-1}(w)$ intersects (and is a subset of) $X^{r,i}$ if and only if $w \in T^{r,i}$. So S^r also satisfies (b) of **(B)**, since each $V(G)_{w'}$ intersects at most one $X^{r,i} \in \mathcal{X}^r$.

So we now show that S^r satisfies **(A)** as well. Each such $V(G)_{w'}$ intersects at most one of the $S^{r,i}$'s in $\mathcal{S}^r$, since each such $V(G)_{w'}$ intersects at most one of the $X^{r,i}$'s in $\mathcal{X}^r$. So the number of vertices in each $V(G)_{w'} \cap S^r$ is at most A, where A is the number of vertices in the largest $S^{r,i} \in \mathcal{S}^r$. By Theorem 3.2, A is $O(C \times B^{1/2})$, where B is the number of vertices in the largest $X^{r,i} \in \mathcal{X}^r$, and C is the maximum, taken over all $X^{r,i} \in \mathcal{X}^r$, of the number of $V(G)_{w'}$'s that intersect $X^{r,i}$.

So $B \le M_1 \times M_2$, where M_1 is the maximum, taken over all $T^{r,i} \in \mathcal{T}^r$, of the number of vertices in $T^{r,i}$, and M_2 is the maximum, taken over all $y \in Y^r$, of the number of vertices in $f_G^{-1}(y)$. We now upper bound M_1. We note the following: if each vertex $y \in T^r$ satisfies $r_1 \le h(y) \le r_2$ for any two integers r_1 and r_2, then each component of T^r has no more than $2^{r_2 - r_1}$ vertices. However, each vertex $y \in Y^r$ satisfies $h(y) - 2\lg y \le r \le h(y)$, and so each $h(y)$ must satisfy $r \le h(y) \le r + 2\lg r + O(1)$. Thus, each component of T^r has $O(2^{2\lg r + O(1)})$ vertices. So M_1 is $O(r^2)$. Next, M_2 is $O(r \times 2^{r/2})$ since each vertex $y \in Y^r$ satisfies $h(y) \le r + 2\lg r + O(1)$ and f satisfies (II). So $B = O(r^3 \times 2^{r/2})$. On the other hand, $C = M_1 \times M_3$, where M_3 is the maximum, taken over all $y \in Y^r$, of the number of vertices in T_y. Each T_y contains $O(h^2(y))$ vertices, where $h(y) \le r + 2\lg r + O(1)$, so $M_3 = O(r^{5/2})$. Thus, $C = O(r^{9/2})$.

So by Theorem 3.2, A is $O(r^6 \times 2^{r/4})$. Now $r \le h(v)$ for each $v \in Y^r$. So $|S^r \cap V(G)_{v'}|$ is $O(h^6(v) \times 2^{\frac{h(v)}{4}})$. Thus, S^r satisfies **(A)**, and Lemma 3.3 follows. ∎

Thus, having proved Lemma 3.3, we now prove Proposition 3.4.

Proposition 3.4. *The resulting mapping $g_G : V(G) \mapsto V(T')$ satisfies both (I') and (II').*

To prove Proposition 3.4, we first show that g_G satisfies (I') via the next three lemmas. Then we show that g_G satisfies (II').

Lemma 3.5. *Let v' and u' be two vertices in T' such that $r(u') = r(v')$, and there is an edge in G with one endpoint in $V(G)_{u'}$ and the other in $V(G)_{v'}$. Then u' and v' are adjacent in T'.*

Proof. We use induction on $r(v')$. There is only one vertex z' in T' such that $r(z') = \lg N$ (namely, z' is the root of T_z, where z is the root of $\hat{T}$), so the conclusion of Lemma 3.5 holds trivially in that case. So, write $r(v') = r(u') = r$. We now assume that **(1)** for all w' and y' such that $r(y') = r(w') \geq r+1$, the conclusion of Lemma 3.5 holds.

Let u and v be the vertices in T such that $u' \in T_u$ and $v' \in T_v$. Then either $u = v$, or u and v are adjacent in T, because $V(G)_{u'}$ and $V(G)_{v'}$ are subsets of $f_G^{-1}(u)$ and $f_G^{-1}(v)$, respectively, and f_G satisfies (I). Thus, $|h(u) - h(v)|$ is no larger than 11.

First suppose that v' is the root of T_v. Then u must be adjacent to v in T. (If $u = v$ then u' would be a descendant of v' in T_v, and then $r(u') < r(v')$.) But then the only way that $r(u')$ could equal $r(v')$ would be if u' were in the top 11 levels of T_u, since $h(u) - h(v) \leq 11$. But then u' and v' are adjacent in T' by (3)(A) of the construction of T'.

So suppose that v' is not the root of T_v, and u' is not the root of T_u. Then $u' = \sigma_{\iota_1}(w')$ and $v' = \sigma_{\iota_2}(y')$ for two vertices w' and y' in T' where $r(w') = r(y') = r+1$. Then $V(G)_{u'} \subseteq V(G_{\iota_1}^{r+1})$, and $V(G)_{v'} \subseteq V(G_{\iota_2}^{r+1})$. So if ι_1 and ι_2 are not the same, then there can be no edge between $V(G)_{u'}$ and $V(G)_{v'}$, because S^{r+1} satisfies **(B)**. So we may assume that $\iota_1 = \iota_2 = \iota$. Then by the induction hypothesis **(1)**, w' and y' are adjacent in T', since $V(G)_{u'} \subseteq V(G)_{w'}$, and $V(G)_{v'} \subseteq V(G)_{y'}$. And by (3) (B) of the construction of T', so are u' and v'. ∎

Lemma 3.6. *Let x and x' be two adjacent vertices in G. Then $|r(g_G(x)) - r(g_G(x'))| \leq 17$.*

Proof. Before we prove this lemma, a remark is in order. Let x be a vertex in C^{r+17} that has not been mapped by g_G to a vertex $w' \in T'$ such that $r(w') > r$. It is not a priori clear that there exists a $v' \in T'$ where $r(v') = r$, and x is in $V(G)_{v'}$. (By Equation (1), this would suffice to prove Lemma 3.5.) We effectively argue below that this must indeed be the case.

Assume that $r(g_G(x)) \le r(g_G(x')) = r+17$. Suppose that the vertex $f_G(x)$, which we will write as u, is in Y^r. Then x will be mapped by g_G to a vertex w' such that $r(w') \ge r$. Indeed, either (A) there exists a vertex $v' \in T_u$ where $r(v') = r$, and $v' \in C^{r+17} \cap V(G)_{v'}$, or (B) there exists a $w' \in T_u$ such that $r(w') > r$, and $g_G(x) = w'$. If (A) happens then $g_G(x) = v'$ by Equation (1). Similarly, suppose u is in $Y^{r'} \setminus Y^r$, for some $r' > r$. Then every vertex u' in T_u satisfies $r(u') > r$. Then $r(g_G(x))$ must be greater than r because $g_G(x)$ is in T_u. So if we prove that $f_G(x)$ is in $Y^{r'}$, for some $r' \ge r$, then Lemma 3.6 will follow.

Now, let y be a vertex in T, and let $\hat{r}$ be any integer where $y \in Y^{\hat{r}}$. Then, if w is one of y's neighbors in T, there is some $\hat{r}'$ where $\hat{r}' \ge \hat{r} - 11$, and $w \in Y^{\hat{r}'}$. (**Proof:** $h(w) \ge h(y) - 11$, because w and y are adjacent in T. Then if we set w' to be the root of T_w, then $r(w') = h(w) \ge h(y) - 11 \ge \hat{r} - 11$, since y is in $Y^{\hat{r}}$ only if $h(y)$ is at least $\hat{r}$. So $r(w') \ge \hat{r} - 11$. So set $\hat{r}' = r(w')$. Then w is in $Y^{\hat{r}'}$ because $w' \in T_w$ satisfies $r(w') = \hat{r}'$.) So, let us write $f_G(x') = y$. Then if $r(g_G(x')) = r+17$, then $y \in Y^{r+17}$, because $g_G(x')$ is in T_y. But $f_G(x) = u$ is adjacent in T to y because f_G satisfies (I). So there does exist an $r' \ge r+6$, where $f_G(x)$ is in $Y^{r'}$. So Lemma 3.6 follows. ∎

We now use Lemmas 3.5 and 3.6 to prove Lemma 3.7, and finish the proof that g_G satisfies (I').

Lemma 3.7. *The function $g_G : V(G) \mapsto V(T')$ satisfies (I').*

Proof. Let x and x' be two adjacent vertices in G, and set $g_G(x) = u'$; $g_G(x') = v'$, and $r(u') \le r(v')$. Next, let u and v be the vertices in T where $v' \in T_v$, and $u' \in T_u$. We may assume that $u' \ne v'$, or we would be done. So we now consider two cases.
Case 1: there is a vertex w' in T_u such that both (1) w' is either an ancestor of u' in T_u, or $w' = u'$, and (2) $r(w') = r(v')$. Then, if w' and v' are distinct, then w' and v' are adjacent in T'. Indeed, $g_G^{-1}(u') \subseteq V(G)_{u'} \subseteq V(G)_{w'}$, and $g_G^{-1}(v') \subseteq V(G)_{v'}$, so there is an edge in G between $V(G)_{w'}$ and $V(G)_{v'}$, and so by Lemma 3.5 there is an edge in T' between v' and w'. However, by Lemma 3.6, $r(v') - r(u') = r(w') - r(u')$ must be no greater than 17, so the distance in T' between u' and w' must be no larger than 17, because u' is w''s descendant in T_u (if u' and w' are distinct). Thus, $d_{T'}(u', v') \le 18$, and so the lemma follows in this case.
Case 2: there is not. Then let w' be the root of T_u; then $r(w') < r(v')$. Then $h(v) - h(u) \le 11$, because v is adjacent to u in T. So v' must be in one of the top 11 levels of T_u because $r(v') > r(w')$. Thus v' and w' must be adjacent in T' by (3) (A) in the construction of T'. However, by Lemma 3.6, the distance

between u' and w' in T' is no more than 16. Thus, the lemma follows in this case as well. ∎

So having shown that g_G satisfies (I'), we next show that g_G satisfies (II'), which states that not too many vertices in G get mapped by g_G to any vertex in T'. Then Proposition 3.4 will follow. Now let v' be an arbitrary vertex in T', and let us write $r(v') = r$. We've already shown, via Lemma 3.3, that $S^r \cap V(G)_{v'}$ is small. So by Equation (1), if $C^{r+17} \cap V(G)_{v'}$ is a small enough set, then g_G satisfies (II'). So, we will effectively establish in Lemma 3.9 that there is a small set S of vertices of $V(G)$, and a small integer d that satisfy the following. For every vertex x in $V(G)_{v'} \cap C^{r+17}$, there must be a path in G of length no more than d from x to a vertex in S. Then, since G has maximum degree 3, the size of $V(G)_{v'} \cap C^{r+17}$ is no larger than $2^{d+2}|S|$, which is small enough to show that g_G satisfies (II').

To prove Lemma 3.9, we will use the following technical lemma.

Lemma 3.8. *Let v' be a vertex in T', and write $r(v') = r$, and let v be the vertex in T such that $v' \in T_v$. Then if v' is an interior vertex in T_v, then the following statements hold.*

(i) *For each vertex $x \in V(G)_{v'} \cap C^{r+17}$, there is a positive integer k, and a path $P = \langle x_0, x_1, \ldots, x_k \rangle$ of $k+1$ vertices, such that $x_k \in S^{r+17k}$, and the x_j's satisfy $r(g_G(x_j)) = r + 17j$ for each $j \le k$.*

(ii) *The integer k in (i) satisfies $k \le (\lg h(v))/3$.*

(iii) *Let x and x' be two vertices in $V(G)_{v'}$. Suppose that, for some integer k, there exist paths $P' = \langle x, x_1, \ldots, x_k \rangle$ and $P = \langle x', x'_1, \ldots, x'_k \rangle$ in G such that $r(g_G(x_j)) = r(g_G(x'_j)) = r + 17j$, for each positive integer $j \le k$. Then there exists a $X^{r+17k,i} \in \mathcal{X}^{r+17k}$, where the $\mathcal{X}^{r'}$'s are as in the proof of Lemma 3.3, such that both x_k and x'_k are in $X^{r+17k,i}$.*

Proof. We prove (i) of Lemma 3.8 first. If x''_j is in $C^{r''+17}$ then x''_j must have a neighbor x''_{j+1} where $r(g_G(x''_{j+1})) = r'' + 17$, by definition of $C^{r''+17}$. So by repeatedly applying Equation (1), we see that there exists a positive integer $k \le (\lg N - r)/17$, and a sequence $x_0, x_1, \ldots, x_k$ of $k+1$ vertices of G, such that, for each nonnegative integer $j < k$, both (i) x_j and x_{j+1} are adjacent in G , and (ii) $r(g_G(x_j)) + 17 = r(g_G(x_{j+1}))$. Furthermore, x_k satisfies either (a) $x_k \in S^{r+17k}$, or (b) $g_G(x_k)$ is a leaf of some T_y.

Suppose that x_k satisfies (b). Then write $f_G(x_{k-1}) = w$ and $f_G(x_k) = y$, and $g_G(x_k) = y'$, and $g_G(x_{k-1}) = w'$. Then $r(y') = r(w') + 17$, and w and y must be the same or adjacent vertices in T, because x_{k-1} and x_k are adjacent in G, and f_G satisfies (I). Thus **(*)** $|h(y) - h(w)| \le 11$. Also, w' and y' are vertices in T_w and T_y, respectively, because g_G maps each vertex in $f_G^{-1}(v)$

to a vertex in T_v for each $v \in T$. But since $r(y') = r(w') + 17$, this implies that the depth of w' in T_w must be at least $17 - h(y) + h(w)$ plus the depth of y' in T_y. However, x_k satisfies (b), so y' is a leaf in T_y. But then the height $2\lg h(w)$ of T_w must be at least $17 - h(y) + h(w)$ plus the height $2\lg h(y)$ of T_y. But it is impossible for $h(w)$ and $h(y)$ to satisfy this inequality and (*) simultaneously. So x_k must satisfy (a), and thus (i) of Lemma 3.8 follows.

We now prove (ii) of Lemma 3.8. Suppose $x \in V(G)_{v'}$ is of distance k in G from a vertex x_k such that $g_G(x_k) = y'$, where $r(y') = r + 17k$. Then set y to be the vertex in T such that $y' \in T_y$. Then x is of distance at most k in G from a vertex in $f_G^{-1}(y)$, because $g_G^{-1}(y')$ is a subset of $f_G^{-1}(y)$. But then $d_T(y, v) \leq k$ because f satisfies (I). Thus y and v satisfy

$$h(v) + 11k \geq h(y), \tag{2}$$

because two vertices w and u are not adjacent in T unless $|h(u) - h(v)| \leq 11$. However,

$$r(v') = r(y') - 17k. \tag{3}$$

But $r(v') \geq h(v) - 2\lg h(v)$. On the other hand, $h(y) \geq r(y')$. So plugging these into Equation (3) gives

$$h(v) - 2\lg h(v) + 17k \leq h(y). \tag{4}$$

So the only way that Inequalities (2) and (4) can be met simultaneously is if $k \leq \lg h(v)/3$. So (2) of this lemma follows.

We now prove (iii) of Lemma 3.8. Let $P_1 = \langle x_k, x_{k-1}, \ldots, x \rangle$ (i.e., the reverse of P) and $P' = \langle x', x'_1, \ldots, x'_k \rangle$ be paths in G from x_k to x, and from x' to x'_k, such that, for each j, both $r(g_G(x_j)) = r(g_G(x'_j)) = r + 17j$. Then $f_G(P_1) = \langle f_G(x_k), \ldots, f_G(x) \rangle$, and $f_G(P') = \langle f_G(x'), \ldots, f_G(x'_k) \rangle$ are walks in T (where one is allowed to stay at the same vertex during any step), because f_G satisfies (I). In fact, because $f_G(x) = f_G(x') = v$, they share a common endpoint; namely, v. So we may concatenate $f_G(P_1)$ and $f_G(P')$ to get another walk $f_G(P_3) = \langle f_G(x'_k), \ldots, f_G(x'_1), f_G(x'), f_G(x), \ldots, f_G(x_k) \rangle$.

Now for any walk W in T, we observe that there is a vertex y covered by W that is an ancestor in $\hat{T}$ of every other vertex in W. Therefore, since $f_G(P_3)$ is a walk in T, there is a vertex y such that $y \in f_G(P_3)$, and y is an ancestor in $\hat{T}$ of every other vertex in $f_G(P_3)$. Then $y \in Y^{r+17k}$. (**Proof:** Because y is covered by $f_G(P_3)$, there is some integer $j \leq k$ such that $y \in \{f_G(x_j), f_G(x'_j)\}$, so let us assume that $f_G(x_j) = y$. Then $g_G(x_j)$ is in T_y. But $r(g_G(x_j)) = r + 17j$. So $y \in Y^{r+17j}$. Then $h(y) - 2\lg h(y) \leq r + 17j$. On the other hand, let $w = f_G(x'_k)$. Then by similar reasoning $w \in Y^{r+17k}$. Thus, $r + 17k \leq h(w)$. But y is w's ancestor in $\hat{T}$ (if y and w are distinct), so

$h(y) \geq h(w)$. So $h(y) - 2\lg h(y) \leq r+17j \leq r+17k \leq h(w) \leq h(y)$. So y must be in Y^{r+17k}.)

So, if we let $u = f_G(x_k)$ and $w = f_G(x'_k)$, then (C) u, w, and y are in the same component of T^{r+17k}. Indeed, for each r', vertices $\tilde{u}$ and $\tilde{v}$ in $V(T^{r'})$ are in the same component of $T^{r'}$ if $\tilde{v}$ is $\tilde{u}$'s ancestor in $\hat{T}$. So (C) follows. So from (C) x_k and x'_k are both in the same $X^{r+17k,i} \in \mathcal{X}^{r+17k}$ by definition of the $X^{r',i}$'s, and so (iii) of Lemma 3.8 follows. ∎

We now use Lemma 3.8 to prove Lemma 3.9.

Lemma 3.9. *Let v' be a vertex in T', and write $r(v') = r$. Then, set v to be the vertex in T_v such that $v' \in T_v$ and set $k = \lg h(v)/3$. Next, for each r, let S^r, $\mathcal{S}^r$ and $\mathcal{X}^r$ be as in the proof of Lemma 3.3. Then there exist k sets $S_1, \ldots, S_k$ in $V(G)$ that satisfy the following.*

(1) $S_j \in \mathcal{S}^{r+17j}$ for each positive integer $j \leq k$.

(2) For each vertex x in $C^{r+17} \cap V(G)_{v'}$, there is an integer $j \leq k$, such that there is a path of length $j+1$ from x to S_j.

Proof. From (i) of Lemma 3.8, we know that, for each vertex $x \in C^{r+17} \cap V(G)_{v'}$, there exists, for at least one positive integer j, a path P_j of length $j+1$, from x to a vertex x'_j in S^{r+17j} such that P_j is of the form given in (i) of Lemma 3.8. From (ii) of Lemma 3.8, we know that $j \leq k$. From (iii) of Lemma 3.8 we know that all such paths P_j of length $j+1$ that start from $C^{r+17} \cap V(G)_{v'}$, and are of the form given by (i) in Lemma 3.8, must also have their other endpoints in the same $X^{r+17j,i} \in \mathcal{X}^{r+17j}$. However, $X^{r+17j,i} \cap S^{r+17j}$ is the set $S^{r+17j,i}$ in $\mathcal{S}^{r+17j}$. Thus, Lemma 3.9 follows. ∎

We are now ready to finish the proof that g_G satisfies (II'). Fix a $v' \in T'$, and let v be the vertex in T such that $v' \in T_v$, and let us write $r(v') = r$, and $h(v) = h$. Suppose that v' is an interior vertex of T_v. Then $g_G^{-1}(v') = (S^r \cup C^{r+17}) \cap V(G)_{v'}$ by Equation (1). However, by Lemma 3.3, $S^r \cap V(G)_{v'}$ has no more than $M' = O(h^9 \times 2^{\frac{h}{4}})$. Since each vertex in G has degree at most 3, it follows from Lemma 3.9 that the number M of vertices in $V(G)_{v'} \cap C^{r+17}$ satisfies

$$M \leq \sum_{j=1}^{\frac{\lg h}{3}} \max_{S \in \mathcal{S}^{r+17j}} 2^{j+2}|S|.$$

However, each $S \in \mathcal{S}^{r+17j}$ has no more than $= O((h+17j)^6 \times 2^{\frac{h+17j}{4}})$ vertices by Lemma 3.3. Thus,

$$M = O(h^8 \times 2^{h/4}).$$

So g_G indeed maps at most $M'+M = O(h^8 \times 2^{\frac{h}{4}})$ vertices of G to v' if v' is an interior vertex of T'.

On the other hand, if v' is a leaf vertex of T_v, then there are at most $O(h^{-2} \times f_G^{-1}(v)) = O(h^{-2} \times 2^{h/2})$ vertices in $V(G)_{v'}$ because $|V(G)_{\sigma_\iota(y')}| \leq |V(G)_{y'}|/2$ for each $\iota \in \{1,2\}$, if y' is an interior vertex in T_v. Thus, g_G satisfies (II') as well as (I'), and Proposition 3.4 follows. ∎

By Lemma 3.1, Theorem 2.1 follows. ∎

4. Small $\mathcal{G}(N,k)$-universal graphs for $k>3$

To construct a $\mathcal{G}(N/4,k)$-universal graph $H(N/4,k)$ of size $O_k(N)$, let T' be as in the construction of $H(N,3)$. Also, let the $V'_{y'}$'s be as in the construction of $H(N,3)$. Then, interconnect every vertex in $V'_{v'}$ with every vertex in $V'_{w'}$ if and only if v' and w' are within distance $36\lg k$ of each other in T'. (Or equivalently, connect every two vertices u'' and v'' in $H(N,3)=H$ if and only if $d_H(u'',v'') \leq 2\lg k$). By a similar line of reasoning that $H(N,3)$ has size $O(N)$, the resulting graph $H(N/4,k)$ has size $O_k(N)$.

We now prove the following theorem.

Theorem 4.1. *The graph $H(N/4,k)$ is $\mathcal{G}(N/4,k)$-universal.*

Proof. Let $G \in \mathcal{G}(N/4,k)$. Then by e.g., replacing each vertex $x \in V(G)$ that has degree d larger than 3 with a complete binary tree K_x with $\lceil d/2 \rceil$ leaves, we see that we may construct a planar graph G' of maximum degree 3 such that there exists an embedding $\rho: V(G) \mapsto V(G')$ such that (1) at most one vertex of G gets mapped to any vertex in G', and (2) if x and $\tilde{x}$ are adjacent vertices in G, then $d_{G'}(\rho(x),\rho(\tilde{x}))$ is no larger than $2\lg k$. Furthermore, since each planar graph on $N/4$ vertices has at most $3N/4$ edges, we may assume that G' has N vertices, and therefore G' is in $\mathcal{G}(N,3)$. Then, by Theorem 2.1, there exists an injective mapping $\chi_{G'}: V(G') \mapsto V(H(N,3))$ such that adjacent vertices in G' are mapped to adjacent vertices in $H(N,3)$. Then the mapping $\chi_{G'} \circ \rho$: $V(G) \mapsto V(H(N,3))$ is injective, and maps adjacent vertices in G to vertices that are of distance $2\lg k$ apart or less in $H(N,3)$. But $H(N/4,k)$ has the same vertex-set as $H(N,3)$, and vertices that are of distance $2\lg k$ or less apart in $H(N,3)$ are connected in $H(N/4,k)$. So $\chi_{G'} \circ \rho$ is an injective mapping from $V(G)$ to $V(H(N/4,k))$ where adjacent vertices in G are mapped to adjacent vertices in $H(N/4,k)$. And so G is indeed isomorphic to a subgraph in $H(N/4,k)$, and thus, Theorem 4.1 follows. ∎

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